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POTATO KUGEL

BY
DOV AHARONOV, M. M. SCHIFFER AND LAWRENCE ZALCMAN

ABSTRACT

Let P be a solid, homogeneous, compact, connected "potato" in space which attracts each point outside it (according to Newton's law) as if all its mass were concentrated at a single point. Answering a question of Lee Rubel, we show that P is a ball. The same conclusion is also obtained under substantially weakened hypotheses.

1. Some time ago, Lee Rubel posed the following problem. Let P be a solid, homogeneous, compact, connected "potato" in space which gravitationally attracts each point outside it as if all its mass were concentrated at a single point, say 0. Must P be spherical, i.e. a ball? In this note we offer an affirmative answer and obtain some extensions.

Surprising connections between potatoes and Peano arithmetic have already been noted in [13]; see also [10, p. 4]. Other points of contact with previous papers, notably [8] and [11], should be fairly evident. It is therefore appropriate to emphasize that our results are altogether unrelated to [7].

2. By assumption, the uniform mass distribution dx on P and the point mass $c\delta_0$ induce identical gravitational fields outside P. Thus [12, p. 4]

$$\int_P \frac{x-y}{|x-y|^3}\,dx = -c\frac{y}{|y|^3}, \qquad y \notin P. \tag{1}$$

The corresponding gravitational potentials differ by at most a constant on each component of the complement of P. Since that complement is, by assumption, connected we have

$$\int_P \frac{dx}{|x-y|} = \frac{c}{|y|} + d, \qquad y \notin P$$

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for an appropriate constant d. Making $|y| \to \infty$ shows that $d = 0$, so that

$$\int_P \frac{dx}{|x-y|} = \frac{c}{|y|}, \qquad y \notin P. \tag{2}$$

The left-hand side of (2), being the convolution of the locally integrable function $1/|x|$ with a bounded measurable function of compact support (*viz.* the characteristic function of P), defines a bounded continuous function *on all of* $\mathbf{R}^3$. It follows that 0 must lie in the interior of P, for otherwise the right-hand side of (2) would be unbounded.

Suppose u is harmonic on (a neighbourhood of) P. We claim that

$$\int_P u(x)dx = cu(0). \tag{3}$$

Indeed, given such a function, we can modify it to be smooth and of compact support, preserving the harmonicity near P. Then [12, p. 13]

$$\begin{aligned}\int_P u(x)dx &= \int_P \left(-\frac{1}{4\pi}\int \frac{\Delta u(y)}{|x-y|}\,dy\right)dx \\ &= -\frac{1}{4\pi}\int \Delta u(y)\left(\int_P \frac{dx}{|x-y|}\right)dy \\ &= -\frac{1}{4\pi}\int \Delta u(y)\frac{c}{|y|}\,dy \\ &= cu(0)\end{aligned}$$

as claimed.

Now choose a point $x_0 \in \partial P$ which lies closest to 0 and take x_n exterior to P, $x_n \to x_0$. The functions

$$v_n(x) = (|x|^2 - |x_n|^2)/|x - x_n|^3$$

are each harmonic on P and tend pointwise to

$$v(x) = (|x|^2 - |x_0|^2)/|x - x_0|^3$$

on $P \setminus \{x_0\}$. Since the $\{v_n\}$ form a uniformly integrable family [6, p. 11], we have

$$\int_P v\,dx = \lim \int_P v_n dx = \lim cv_n(0) = cv(0)$$

so v satisfies (3).

The rest of the argument follows [4]. Set

$$U(x) = 1 + |x_0|\,v(x)$$

and let B be the ball centered at 0 having radius $|x_0|$. Then $B \subset P$ and

$$0 = cU(0) = \int_P U dx = \int_B U dx + \int_{P \setminus B} U dx$$

$$= 0 + \int_{P \setminus B} U dx.$$

Since $U(x) > 0$ for $|x| > |x_0|$ and P is assumed to be solid (so that $\overline{P^0} = P$), we obtain a contradiction unless $P \subset B$. Thus $P = B$ as required.

As usual, an analogous result obtains in $\mathbf{R}^n$, with the kernel $|x-y|^{-1}$ replaced by $|x-y|^{2-n}$ when $n \geqq 4$ and by $-\log|x-y|$ for $n = 2$. In the sequel we shall continue to state our results for potatoes in $\mathbf{R}^3$ with the understanding that similar results hold, *mutatis mutandis*, in spaces of arbitrary dimension.

It is worth noting that the condition (2) can be relaxed very considerably. It is enough, for instance, to assume that (2) holds only for y sufficiently large (or, indeed, for y belonging to any open set in the complement of P). For both sides of (2) are harmonic off P, so the full force of (2) follows by harmonic continuation. Observe also that the constant c in (2) is the volume of P. This is physically obvious and, in any case, follows by multiplying both sides of (2) by $|y|$ and making $y \to \infty$.

To what extent does the result of this section mirror a more general phenomenon? In other words, is the map

$$\text{pot} : \text{POT} \to \text{Pot},$$

which associates to each object in the (discrete) category POT of potatoes its external potential or, better, the corresponding object in the category Pot of (germs of) potentials at infinity, a monofunctor? Obviously, an affirmative solution would constitute a notable extension of the result obtained above. Unfortunately, however, the answer is negative. When $n = 2$, this follows from an example due to Sakai [9]. As pointed out to us by B. Weiss, Sakai's example can be easily modified to yield examples in higher dimensions as well.

3. In the remaining sections we shall be concerned with extending the result of the preceding paragraph by relaxing the various hypotheses (connectedness, compactness, homogeneity, and solidity) initially placed on our potato. On physical grounds, we shall continue to assume that P is both connected and solid (i.e., $P = \overline{P^0}$, the closure of its interior, and $\mathbf{R}^3 \setminus P$ is connected).

Actually, an amusing aspect of the previous proof is that the assumption of connectedness played no role whatsoever and could accordingly be dropped.

This suggests consideration of the more physically meaningful configuration of a "sack of potatoes", i.e. a finite collection of disjoint (connected) potatoes.

Accordingly, let $S = \bigcup_{j=1}^{l} P_j$, where the P_j are pairwise disjoint potatoes. Suppose there exist points $x_1, x_2, \cdots, x_l$ in $\mathbf{R}^3$ and scalars $a_1, a_2, \cdots, a_l$ such that

$$\int_S \frac{dx}{|x-y|} = \sum_{j=1}^{l} \frac{a_j}{|y-x_j|}, \qquad y \notin S. \tag{4}$$

Arguing as before, we see that if $a_j \neq 0$, x_j must lie in the interior of S and that

$$\int_S u(x)dx = \sum_{j=1}^{l} a_j u(x_j) \tag{5}$$

for any function u harmonic in a neighbourhood of S. Suppose some component P_j of S fails to meet the set $X = \{x_1, \cdots, x_l\}$. Choosing u to be 1 on a neighbourhood of P_j and 0 elsewhere, we obtain a contradiction to (5). Thus each P_j contains at least, and hence exactly, one point of X. Renumbering if necessary, we have $x_j \in P_j$. We may now take u in (5) to be an arbitrary function harmonic in a neighbourhood of P_j and vanishing on a neighbourhood of the rest of S. Then (5) becomes

$$\int_{P_j} u(x)dx = a_j u(x_j)$$

which is, essentially, (3). The situation is thus reduced to that of a single potato, and the previous argument applies. It follows that S is a collection of disjoint balls

$$P_j = \{|x - x_j| \leqq r_j\}, \qquad \text{where } r_j = \sqrt[3]{3a_j/4\pi}.$$

Conclusions are possible also when the number of points in X exceeds the number of components of S. In this case, the points of X are partitioned among the potatoes in S, and each potato satisfies an appropriate quadrature identity [1] with respect to the points of X it contains. Detailed discussion of the geometric consequences of this fact is deferred to a later occasion.

4. Only slight variations are required to adapt the argument of §2 to the case of an unbounded potato. Suppose, for instance, that the integral $\int_P |x|^{-1}dx$ is finite, so that the potential in (2) exists. Fixing y exterior to P, we have by (2)

$$\int_P \left[\frac{1}{|x-\rho y|} - \frac{1}{|\rho x - y|}\right] dx = \frac{c}{\rho|y|} - \frac{c}{|y|} \tag{6}$$

for $\rho > 0$ sufficiently close to 1. Differentiating (6) with respect to ρ and setting $\rho = 1$, we obtain

$$\int_P \frac{|x|^2 - |y|^2}{|x-y|^3}\, dx = -\frac{c}{|y|}.$$

We may now let y tend to x_0 and complete the argument as before. This reasoning also provides an attractive alternate route to the result of §2.

Of course, Rubel's question makes sense even if the potential fails to converge, so long as the field of (1) remains finite. In that case, different techniques seem required to settle the issue.

5. One cannot hope to dispense entirely with the hypothesis that P is homogeneous. Indeed, any measure for which the analogue of (3) holds will obviously satisfy the analogue of (2); and, on *any* potato, such measures exist in abundance. On the other hand, consideration of rotation-invariant measures allows us to weaken this hypothesis very considerably. This is evident already from the reasoning in §2, which applies *verbatim* if the uniform mass distribution is replaced by $f(|x|)dx$, where f is a bounded positive function. Elaboration of the argument in [3] enables us to go further and consider mass distributions of variable sign. We turn directly to the details.

Accordingly, let $P \subset \mathbf{R}^3$ be a compact, connected potato containing 0 and let $f = f(|x|)$ be a smooth radial function, defined in a neighbourhood of P, *which does not vanish identically on any open set which intersects* ∂P. We call a function with this property *admissible* for P. Set

$$\psi(y) = \int_P \frac{f(|x|)}{|x-y|}\, dx.$$

Then $\psi \in C^1(\mathbf{R}^3)$ since

$$\nabla\psi(y) = -\int_P f(|x|)\frac{x-y}{|x-y|^3}\, dx,$$

the (vector-valued) convolution of a bounded measurable function of compact support with the locally integrable function $x/|x|^3$. Suppose that the force field induced by f on the exterior of P is Newtonian, i.e.

$$\nabla\psi(y) = -c\frac{y}{|y|^3}, \qquad y \notin P. \tag{7}$$

We claim that P is a ball centered at 0.

It suffices to show that ψ is a radial function on P, i.e. $\psi(x) = \psi(r)$ where $r = |x|$, $x \in P$. Indeed by (7) and the continuity of $\nabla\psi$, we have

(8) $$\nabla\psi(x) = \psi'(r)\frac{x}{r} = -c\frac{x}{r^3}, \qquad x \in \partial P$$

so

(9) $$\psi'(r) = -\frac{c}{r^2}$$

on ∂P. If P is not a ball, we can find $x_1, x_2 \in \partial P$ with $r_1 = |x_1| < |x_2| = r_2$. The open shell $S = \{r_1 < |x| < r_2\}$ then intersects both ∂P and P^0. Since ψ is radial on P we have, by (9),

$$\nabla^2\psi(x) = \psi''(r) + \frac{2}{r}\psi'(r) = 0, \qquad x \in S \cap P^0.$$

But ([12, p. 16])

(10) $$\nabla^2\psi(x) = -4\pi f(x), \qquad x \in P^0.$$

Thus $f(x) = 0$ for $x \in S \cap P^0$, and this contradicts the assumption that f is admissible for P.

The proof that ψ is radial on P hinges on the following lemma, which may be of some interest in itself.

LEMMA. *Let $D \subset \mathbf{R}^n$ be a bounded domain containing 0 and let $u \in C^1(\bar{D})$ be harmonic on D. Assume that the gradient vector ∇u is parallel to the radius vector at each point of ∂D, i.e.*

(11) $$\nabla u(x) = \lambda(x)x, \qquad x \in \partial D.$$

Then u is constant on D.

PROOF. Set

$$g(x) = g_{ij}(x) = x_i\frac{\partial u}{\partial x_j} - x_j\frac{\partial u}{\partial x_i}, \qquad 1 \leqq i < j \leqq n.$$

A simple calculation shows that $\nabla^2 g = 0$ on D, and by (11) $g = 0$ on ∂D. Thus, $g = g_{ij}$ vanishes identically on D for each pair of indices i, j. It follows that $\nabla u(x) = \lambda(x)x$ throughout D, i.e. ∇u is parallel to the radius vector at each point of D. Let S be a sphere contained in D, centered at 0. At each point $x \in S$, S is orthogonal to the radius vector at x and hence to $\nabla u(x)$. Thus, at each point of S the directional derivatives of u vanish in all directions tangent to S. It follows that u is constant on S, so u is a function of $|x|$ near the origin. Thus

$$u(x)=\begin{cases} a|x|^{2-n}+c, & n\geqq 3\\ a\log|x|+c, & n=2\end{cases}$$

for x sufficiently small. Since u is not singular at 0, $a=0$ and $u(x)=c$ for small x. By continuation, $u(x)=c$ throughout D.

Returning to the proof that ψ is radial, we set

$$\varphi(y)=\int_B \frac{f(|x|)}{|x-y|}\,dx,$$

where B is a ball centered at the origin which contains P. This function is obviously radial and continuously differentiable on $\mathbf{R}^3$; thus

$$\nabla\varphi(x)=\frac{\varphi'(r)}{r}\,x,\qquad r=|x| \tag{12}$$

for *all* $x\neq 0$ and hence, in particular, for $x\in\partial P$. Moreover,

$$\nabla^2\varphi(x)=-4\pi f(x),\qquad x\in B^0. \tag{13}$$

It follows from (10), (13), (8), and (12) that $\psi-\varphi$ satisfies the hypotheses of the lemma on $D=P^0$. Hence $\psi(x)=\varphi(x)+c$ throughout P, and ψ is radial on P as claimed.

The regularity assumptions on f can be relaxed considerably. It suffices, for instance, to assume that f is a bounded measurable function. Equations (10) and (13) then hold in the sense of distributions [12, p. 18], and the harmonicity of $\psi-\varphi$ follows from Weyl's lemma.

The argument above also applies to the case of a sack of potatoes, considered in §3. Indeed, suppose we are given disjoint potatoes $P_1, P_2,\cdots, P_l$ each endowed with a (not necessarily positive) mass distribution f_j which is radial with respect to some fixed point $x_j\in P_j$ and which is admissible for P_j. Let ψ be the corresponding potential and suppose that

$$\psi(y)=\sum_{j=1}^{l}\frac{c_j}{|y-x_j|},\qquad y\notin\bigcup_{j=1}^{l}P_j.$$

Then for fixed k, the function

$$\psi_k(y)=\psi(y)-\sum_{j\neq k}\frac{c_j}{|y-x_j|}$$

satisfies

$$\nabla^2\psi_k(x) = -4\pi f_k(\ x - x_k\ ), \qquad x \in P_k^0,$$

$$\nabla\psi_k(x) = -\frac{c_k}{x - x_k{}^3}(x - x_k), \qquad x \in \partial P_k,$$

and we may argue exactly as before to conclude that P_k is a ball centered at x_k.

Finally, let us note that the argument of this section is easily adapted to handle the case of potatoes "with holes", i.e. for which $\tilde{P} = \mathbf{R}^3 \setminus P$ is not connected. Indeed, suppose f is admissible for $\tilde{P}$ and that (7) holds; we allow different values of c on different components of P and require, for physical reasons, that $c = 0$ on that component (if any) which contains 0. The reasoning used in the lemma shows that ψ is constant on each component of $S \cap P$ for any sphere S about the origin. It then follows as before that each component of ∂P is a sphere about the origin. Since P is itself connected, it can be only a ball or a spherical shell. Details are left to the reader.

6. *Dessert.* See [2] and [5].

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TECHNION — ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA, ISRAEL

STANFORD UNIVERSITY
STANFORD, CA 94305 USA

UNIVERSITY OF MARYLAND
COLLEGE PARK, MD 20742 USA